Exercises for 2.8 and 2.9

1. Take the procedure for enumerating class-1 sentences. The procedure lists sentences in groups of five (five sentences for each fraction). List the first 3 groups of sentences that the procedure lists (that's 15 sentences in total).

   Answer Key

   Let $S_1, S_2, S_3, \ldots$ be the sentences of class-1. Then:

   The first three fractions enumerated on Figure 2.1 are 1/1, 2/1, 1/2. We plug the numerator into n and the denominator into m. So the first three groups of sentences are:

   $S_1, \neg S_1, S_1 \wedge S_1, S_1 \vee S_1, S_1 \supset S_1,\ S_2, \neg S_2, S_2 \wedge S_1, S_2 \vee S_1, S_2 \supset S_1,$ $S_1, \neg S_1, S_1 \wedge S_2, S_1 \vee S_2, S_1 \supset S_2$

2. The procedure for enumerating class-1 sentences lists sentences in groups of five. One of them is $A_n \wedge A_m$. Why is it ok not to list $A_m \wedge A_n$?

   Answer Key

   $A_n \wedge A_m$ is listed when we encounter n/m, and we know $A_m \wedge A_n$ will also be listed eventually because for every fraction n/m, the fraction m/n will also be listed eventually.

   Quite a few of you answered that it is because $A_m \wedge A_n$ is logically equivalent to $A_n \wedge A_m$. But that wouldn't explain why it is also ok not to list $A_m \supset A_n$.

3. "Even a formal language with infinitely many atomic sentences could not express all the ways the world could be." Is this true or false? Explain.

   Answer Key

   Each interpretation represents a way the world could be. Given infinitely many atomic sentences, there are more interpretations than there are natural numbers. But even with infinitely many atomic sentences, there are only as many sentences as there are natural numbers. So the formal language cannot express all the ways the world could be.

   Any language whose sentences are composed of countably many characters has only as many sentences as there are natural numbers. In particular, if a language can be presented on a computer screen, it only has as many sentences as there are natural numbers—because a computer must represent a sentence by a natural number. Now consider this:

For any subset S of natural numbers, there is a fact whether or not the number 1 is in that subset. Could we express for each subset S of natural numbers the claim that the number 1 is a member of S? The answer is No: there are more subsets of natural numbers than there are natural numbers, but only as many sentences as there are natural numbers. So there are uncountably many very simple mathematical facts that no language—formal or otherwise—of ours can even express.

And if they can't be expressed, they can't be proven either. If you are interested in mathematics, this is a point worth letting sink in. Just to repeat the point: no matter how powerful our mathematical language, there are uncountably many subsets of natural numbers of which we cannot even ask whether 1 is a member of that set. And once you see the point, you will notice that there many more mathematical facts we cannot even express, let alone prove.