Exercises for 3.13–3.14

1. Here is something obvious. If we have evidence that $P \vee Q$ and we have evidence that $\neg P$, we have evidence that Q. Our proof system confirms this. Add the missing datums in the following derivation:

1.	Γ	$\vdash P \vee Q$	premise
2.	Δ	$\vdash \neg P$	premise
3.	P	$\vdash P$	A
4.	$\underline{\Delta, \neg Q}$	$\vdash \neg P$	2
5.	$\underline{P, \neg Q}$	$\vdash P$	3
6.	$\underline{\Delta, P}$	$\vdash \neg\neg Q$	4,5,¬I
7.	Δ, P	$\vdash Q$	6,¬E
8.	Q	$\vdash Q$	A
9.	$\underline{\Gamma, \Delta}$	$\vdash Q$	1,7,8,∨E

2. Derivations can often be adapted to prove something similar. For instance, the above can be adapted easily to derive from $\Gamma \vdash \neg P \vee Q$ and $\Delta \vdash P$ to $\Gamma, \Delta \vdash Q$. Fill in the missing datums and annotations.

1.	Γ	$\vdash \neg P \vee Q$	premise
2.	Δ	$\vdash P$	premise
3.	$\underline{\neg P}$	$\vdash \neg P$	A
4.	$\underline{\Delta, \neg Q}$	$\vdash P$	2
5.	$\underline{\neg P, \neg Q}$	$\vdash \neg P$	3
6.	$\underline{\Delta, \neg P}$	$\vdash \neg\neg Q$	4,5,¬I
7.	$\underline{\Delta, \neg P}$	$\vdash Q$	6,¬E
8.	Q	$\vdash Q$	A
9.	$\underline{\Gamma, \Delta}$	$\vdash Q$	1,7,8,∨E

3. The Greek capital letters on the datum side are place-holders. You can plug anything you want into them. Take the derivation in Problem 2 above. You can plug P into Δ and add one more step at the end to show that you can infer from $\Gamma \vdash \neg P \vee Q$ to $\Gamma \vdash P \supset Q$ (which we should expect given the way the conditional is defined). Construct such a derivation.

Answer Key

1.	Γ	$\vdash \neg P \vee Q$	premise
2.	P	$\vdash P$	A
3.	$\neg P$	$\vdash \neg P$	A
4.	$P, \neg Q$	$\vdash P$	2
5.	$\neg P, \neg Q$	$\vdash \neg P$	3
6.	$P, \neg P$	$\vdash \neg\neg Q$	4,5,¬I
7.	$P, \neg P$	$\vdash Q$	6,¬E
8.	Q	$\vdash Q$	A
9.	Γ, P	$\vdash Q$	1,7,8,∨E
10.	Γ	$\vdash P \supset Q$	9,⊃I

4. The following is a derivation from $\Gamma \vdash P \supset Q$ to $\Gamma \vdash \neg P \vee Q$. Again, we should expect that there is such a derivation given the way the conditional was defined. Fill in the missing parts of the derivation:

1.	Γ	$\vdash P \supset Q$	premise
2.	$\neg(\neg P \vee Q)$	$\vdash \underline{\neg(\neg P \vee Q)}$	A
3.	$\underline{\neg P}$	$\vdash \neg P$	A
4.	$\neg P$	$\vdash \neg P \vee Q$	$\underline{\text{3,∨I}}$
5.	$\underline{\neg(\neg P \vee Q), \neg P}$	$\vdash \underline{\neg(\neg P \vee Q)}$	2
6.	$\neg(\neg P \vee Q)$	$\vdash \neg\neg P$	4,5,¬I
7.	$\underline{\neg(\neg P \vee Q)}$	$\vdash \underline{P}$	6,¬E
8.	$\underline{\Gamma, \neg(\neg P \vee Q)}$	$\vdash \underline{Q}$	1,7,⊃E
9.	$\Gamma, \neg(\neg P \vee Q)$	$\vdash \neg P \vee Q$	8,∨I
10.	Γ	$\vdash \neg\neg(\neg P \vee Q)$	$\underline{\text{2,9,¬I}}$

11. Γ $\vdash \neg P \vee Q$ .... 10,¬E

5. Suppose there is evidence that $P \supset Q$. In that case, there is is evidence that $\neg Q \supset \neg P$. Our proof system confirms this. Construct a derivation from $\Gamma \vdash P \supset Q$ to $\Gamma \vdash \neg Q \supset \neg P$. Hint: you can adapt and modify one the derivations given in Section 3.13.

Answer Key

1. Γ $\vdash P \supset Q$ .... premise
2. $\neg Q$ $\vdash \neg Q$ .... A
3. P $\vdash P$ .... A
4. Γ, P $\vdash Q$ .... 1,3,⊃E
5. $\neg Q, P$ $\vdash \neg Q$ 2
6. $\Gamma, \neg Q$ $\vdash \neg P$ .... 4,5,¬I
7. Γ $\vdash \neg Q \supset \neg P$ .... 6,⊃I

6. Correct any errors in the annotations of the following derivation from $\Gamma \vdash P \supset Q$ and $\Delta \vdash R \vee \neg Q$ to $\Gamma, \Delta \vdash P \supset R$ (the only errors are in the annotations):

1. Γ $\vdash P \supset Q$ .... premise
2. Δ $\vdash R \vee \neg Q$ ....

(premise)

3. R $\vdash R$ .... A
4. R, P $\vdash R$ 3
5. R $\vdash P \supset R$ ....

(4,)

⊃I

6. $\neg Q$ $\vdash \neg Q$ ....

Ⓐ

7. $P \quad \vdash P$ .... A

8. $\Gamma, P \quad \vdash Q$ 1,7,

(⊃E)

9. $\neg Q, P \quad \vdash \neg Q$ 6

10. $\Gamma, \neg Q \quad \vdash \neg P$ ....

(8,9)

,¬I

11. $\Gamma, \neg Q \quad \vdash \neg P \vee R$ ....

(10)

,∨I

12. $\neg P \quad \vdash \neg P$ .... A

13. $\neg P, \neg R \quad \vdash \neg P$ ....

(12)

14. $P, \neg R \quad \vdash P$ ....

(7)

15. $\neg P, P \quad \vdash \neg\neg R$ 13,14,

(¬I)

16. $\neg P, P \quad \vdash R$ 15,

(¬E)

17. $\neg P \quad \vdash P \supset R$ 16,

$\supset$I

18. $\Gamma, \neg Q \vdash P \supset R$ .... 11,17,5,$\vee$E
19. $\Gamma, \Delta \vdash P \supset R$ ....

2,5,18

,$\vee$E

7. If you have evidence that $(W \vee Y) \supset Z$, you have evidence that $W \supset Z$. Construct a derivation from $\Gamma \vdash (W \vee Y) \supset Z$ to $\Gamma \vdash W \supset Z$.

Answer Key

1. $\Gamma \vdash (W \vee Y) \supset Z$ .... premise
2. $W \vdash W$ .... A
3. $W \vdash W \vee Y$ .... 2,$\vee$I
4. $\Gamma, W \vdash Z$ .... 1,3,$\supset$E
5. $\Gamma \vdash W \supset Z$ .... 4,$\supset$I