Exercises for 5.12–5.14

1. Add missing items:

    1. $\forall xFx \wedge \forall xGx \vdash \forall xFx \wedge \forall xGx$ .... __
    2. $\forall xFx \wedge \forall xGx \vdash \forall xFx$ .... __
    3. $\forall xFx \wedge \forall xGx \vdash Fa$ .... __
    4. $\forall xFx \wedge \forall xGx \vdash \forall xGx$ .... __
    5. $\forall xFx \wedge \forall xGx \vdash Gb$ .... __
    6. $\forall xFx \wedge \forall xGx \vdash Fa \wedge Gb$ .... __
    7. $\forall xFx \wedge \forall xGx \vdash \forall y(Fa \wedge Gy)$ .... __
    8. $\forall xFx \wedge \forall xGx \vdash \forall x\forall y(Fx \wedge Gy)$ .... __

2. Prove $\forall x(Fx \wedge \forall yGy) \vdash \forall x\forall y(Fx \wedge Gy)$. Hint: compare with proof above. ....
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3. Add missing items:

1.	$\forall xFx \vee \forall xGx$	$\vdash$	$\forall xFx \vee \forall xGx$	. . . . A
2.	$\forall xFx$	$\vdash$	$\forall xFx$	. . . . A
3.	$\forall xFx$	$\vdash$	__	. . . . 2,∀E
4.	$\forall xFx$	$\vdash$	__	. . . . 3,∨I
5.	__	$\vdash$	__	. . . . A
6.	__	$\vdash$	Gb	. . . . __
7.	__	$\vdash$	__	. . . . __
8.	$\forall xFx \vee \forall xGx$	$\vdash$	$Fa \vee Gb$	. . . . 1,4,7,∨E
9.	$\forall xFx \vee \forall xGx$	$\vdash$	$\forall y(Fa \vee Gy)$	. . . . __
10.	$\forall xFx \vee \forall xGx$	$\vdash$	$\forall x \forall y(Fx \vee Gy)$	. . . . __

4. Prove $\forall x(Fx \vee \forall yGy) \vdash \forall x \forall y(Fx \vee Gy)$.

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5. Prove $\forall x Fx \wedge \exists x Gx \vdash \forall x \exists y(Fx \wedge Gy)$. Here are the first few lines:

    1. $\forall x Fx \wedge \exists x Gx \ \vdash \ \forall x Fx \wedge \exists x Gx$ ....A
    2. $\forall x Fx \wedge \exists x Gx \ \vdash \ \forall x Fx$ .... 1,∧E
    3. $\forall x Fx \wedge \exists x Gx \ \vdash \ Fa$ .... 2,∀E
    4. $\forall x Fx \wedge \exists x Gx \ \vdash \ \exists x Gx$ .... 1,∧E
    5. $Gb \ \vdash \ Gb$ ....A

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6. Prove $\forall x Fx \wedge \neg\forall x Gx \vdash \forall x \exists y(Fx \wedge \neg Gy)$. Hint: Use Quantifer Exchange to get rid of the negation in front of the universal quantifier; you will get something that looks very much like one of the problems above.

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7. Here is something obvious: if there are at least two Fs, there is at least one F. Let's make sure that our way of counting delivers this result. Prove $\exists x[Fx \wedge \exists y(Gy \wedge x{\neq}y)] \vdash \exists x Fx$. Hint: for how to do the first few lines, consider the next problem.

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8. Prove $\exists x(Fx \wedge \exists y Gy) \vdash \exists x \exists y(Fx \wedge Gy)$. Here are the first few lines.

1.	$\exists x(Fx \wedge \exists y Gy)$	$\vdash \exists x(Fx \wedge \exists y Gy)$	. . . . A
2.	$Fa \wedge \exists y Gy$	$\vdash Fa \wedge \exists y Gy$	. . . . A
3.	$Fa \wedge \exists y Gy$	$\vdash Fa$	. . . . 2,∧E
4.	$Fa \wedge \exists y Gy$	$\vdash \exists y Gy$	. . . . 2,∧E

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9. In Section 5.14 I claim without explanation that formulas 4 and 5 say the same thing. If I am right, we expect a derivation from formula 4 to formula 5. Here is a such a derivation. Add missing items (note: $\neg(a{\neq}b)$ is the same as $\neg\neg(a{=}b)$):

   To aid visibility, let

   Γ: $\exists x[Fx \wedge \neg\exists y(Fy \wedge x{\neq}y)]$

   Δ: $Fa \wedge \neg\exists y(Fy \wedge a{\neq}y)]$

   Notice Δ has a constant so we need to be mindful of that.

1.	Γ	$\vdash \exists x[Fx \wedge \neg\exists y(Fy \wedge x{\neq}y)]$	A
2.	Δ	$\vdash Fa \wedge \neg\exists y(Fy \wedge a{\neq}y)$	A
3.	Δ	$\vdash$ __	2, $\wedge$E
4.		$\vdash$ __	QE
5.	Δ	$\vdash \forall y\neg(Fy \wedge a{\neq}y)$	3,4,$\supset$E
6.	Δ	$\vdash$ __	5,$\forall$E
7.		$\vdash \neg(Fb \wedge a{\neq}b) \supset [\neg Fb \vee \neg(a{\neq}b)]$	DM
8.	Δ	$\vdash$ __	6,7,$\supset$E
9.		$\vdash [\neg Fb \vee \neg(a{\neq}b)] \supset [\neg\neg Fb \supset \neg(a{\neq}b)]$	EL
10.	__	$\vdash$ __	8, 9, $\supset$E
11.	Fb	$\vdash Fb$	A
12.		$\vdash Fb \supset \neg\neg Fb$	DN
13.	Fb	$\vdash \neg\neg Fb$	__
14.	Δ, Fb	$\vdash \neg(a{\neq}b)$	10,13,$\supset$E
15.	Δ, Fb	$\vdash a{=}b$	__
16.	__	$\vdash$ __	15,$\supset$I
17.	Δ	$\vdash$ __	16,$\forall$I
18.	Δ	$\vdash$ __	__
19.	Δ	$\vdash Fa \wedge \forall y(Fy \supset a{=}y)$	17,18,$\wedge$I
20.	__	$\vdash \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$	19,$\exists$I
21.	Γ	$\vdash \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$	__

10. Here is another way of saying that there is exactly one thing:

    $\exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$

    Prove $\exists x[Fx \wedge \forall y(Fy \supset x{=}y)] \vdash \exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$. Here are the first two lines:

    1. $\exists x[Fx \wedge \forall y(Fy \supset x{=}y)] \quad \vdash \quad \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$. A
    2. $Fa \wedge \forall y(Fy \supset a{=}y) \quad \vdash \quad Fa \wedge \forall y(Fy \supset a{=}y)$ . . . . . . A

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11. Prove $\exists x \forall y(Fx \wedge Gy) \vdash \exists x(Fx \wedge \forall y Gy)$. (This is a simplified version of the reverse of the above).

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