Exercises for 5.12–5.14

1. Add missing items:

    1. $\forall xFx \land \forall xGx \vdash \forall xFx \land \forall xGx$ .... A
    2. $\forall xFx \land \forall xGx \vdash \forall xFx$ .... 1,∧E
    3. $\forall xFx \land \forall xGx \vdash Fa$ .... 2,∀E
    4. $\forall xFx \land \forall xGx \vdash \forall xGx$ .... 1,∧E
    5. $\forall xFx \land \forall xGx \vdash Gb$ .... 4,∀E
    6. $\forall xFx \land \forall xGx \vdash Fa \land Gb$ .... 3,5,∧I
    7. $\forall xFx \land \forall xGx \vdash \forall y(Fa \land Gy)$ .... 6, ∀I
    8. $\forall xFx \land \forall xGx \vdash \forall x\forall y(Fx \land Gy)$ .... 7,∀I

2. Prove $\forall x(Fx \land \forall yGy) \vdash \forall x\forall y(Fx \land Gy)$. Hint: compare with proof above.

    1. $\forall x(Fx \land \forall yGy) \vdash \forall x(Fx \land \forall yGy)$ .... A
    2. $\forall x(Fx \land \forall yGy) \vdash Fa \land \forall yGy$ .... 1,∀E
    3. $\forall x(Fx \land \forall yGy) \vdash Fa$ .... 2,∧E
    4. $\forall x(Fx \land \forall yGy) \vdash \forall yGy$ .... 2,∧E
    5. $\forall x(Fx \land \forall yGy) \vdash Gb$ .... 4,∀E
    6. $\forall x(Fx \land \forall yGy) \vdash Fa \land Gb$ .... 3,5,∧I
    7. $\forall x(Fx \land \forall yGy) \vdash \forall y(Fa \land Gy)$ .... 6,∀I
    8. $\forall x(Fx \land \forall yGy) \vdash \forall x\forall y(Fx \land Gy)$ .... 7,∀I

3. Add missing items:

    1. $\forall xFx \lor \forall xGx \vdash \forall xFx \lor \forall xGx$ . . . . A
    2. $\forall xFx \vdash \forall xFx$ . . . . A
    3. $\forall xFx \vdash \underline{Fa}$ . . . . 2,∀E
    4. $\forall xFx \vdash \underline{Fa \lor Gb}$ . . . . 3,∨I
    5. $\underline{\forall xGx} \vdash \underline{\forall xGx}$ . . . . A
    6. $\underline{\forall xGx} \vdash Gb$ . . . . $\underline{\text{5,}\forall\text{E}}$
    7. $\underline{\forall xGx} \vdash \underline{Fa \lor Gb}$ . . . . $\underline{\text{6,}\lor\text{I}}$
    8. $\forall xFx \lor \forall xGx \vdash Fa \lor Gb$ . . . . 1,4,7,∨E
    9. $\forall xFx \lor \forall xGx \vdash \forall y(Fa \lor Gy)$ . . . . $\underline{\text{8, }\forall\text{I}}$
    10. $\forall xFx \lor \forall xGx \vdash \forall x\forall y(Fx \lor Gy)$ . . . . $\underline{\text{9,}\forall\text{I}}$

4. Prove $\forall x(Fx \lor \forall yGy) \vdash \forall x\forall y(Fx \lor Gy)$.

    1. $\forall x(Fx \lor \forall yGy) \vdash \forall x(Fx \lor \forall yGy)$ . . . . A
    2. $\forall x(Fx \lor \forall yGy) \vdash Fa \lor \forall yGy$ . . . . 1,∀E
    3. $Fa \vdash Fa$ . . . . A
    4. $Fa \vdash Fa \lor Gb$ . . . . 3,∨I
    5. $\forall yGy \vdash \forall yGy$ . . . . A
    6. $\forall yGy \vdash Gb$ . . . . 5,∀E
    7. $\forall yGy \vdash Fa \lor Gb$ . . . . 6,∨I
    8. $\forall x(Fx \lor \forall yGy) \vdash Fa \lor Gb$ . . . . 2,4,7,∨E
    9. $\forall x(Fx \lor \forall yGy) \vdash \forall y(Fa \lor Gy)$ . . . . 8,∀I
    10. $\forall x(Fx \lor \forall yGy) \vdash \forall x\forall y(Fx \lor Gy)$ . . . . 9,∀I

5. Prove $\forall xFx \wedge \exists xGx \vdash \forall x\exists y(Fx \wedge Gy)$. Here are the first few lines:

1.	$\forall xFx \wedge \exists xGx$	$\vdash$	$\forall xFx \wedge \exists xGx$	......A
2.	$\forall xFx \wedge \exists xGx$	$\vdash$	$\forall xFx$	......1,∧E
3.	$\forall xFx \wedge \exists xGx$	$\vdash$	Fa	......2,∀E
4.	$\forall xFx \wedge \exists xGx$	$\vdash$	$\exists xGx$	......1,∧E
5.	Gb	$\vdash$	Gb	......A

6.	$\forall xFx \wedge \exists xGx, Gb$	$\vdash$	$Fa \wedge Gb$	......3,5, ∧I
7.	$\forall xFx \wedge \exists xGx, Gb$	$\vdash$	$\exists y(Fa \wedge Gy)$	......6,∃I
8.	$\forall xFx \wedge \exists xGx, Gb$	$\vdash$	$\forall x\exists y(Fx \wedge Gy)$	......7,∀I
9.	$\forall xFx \wedge \exists xGx$	$\vdash$	$\forall x\exists y(Fx \wedge Gy)$	......4,8,∃E

6. Prove $\forall xFx \wedge \neg\forall xGx \vdash \forall x\exists y(Fx \wedge \neg Gy)$. Hint: Use Quantifer Exchange to get rid of the negation in front of the universal quantifier; you will get something that looks very much like one of the problems above.

1.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	$\forall xFx \wedge \neg\forall xGx$	......A
2.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	$\neg\forall xGx$	......1,∧E
3.		$\vdash$	$\neg\forall xGx \supset \exists x\neg Gx$	......QE
4.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	$\exists x\neg Gx$	......2,3,⊃E
5.	$\neg Gb$	$\vdash$	$\neg Gb$	......A
6.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	$\forall xFx$	......1,∧E
7.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	Fa	......6,∀E
8.	$\forall xFx \wedge \neg\forall xGx, \neg Gb$	$\vdash$	$Fa \wedge \neg Gb$	......5,7,∧I
9.	$\forall xFx \wedge \neg\forall xGx, \neg Gb$	$\vdash$	$\exists y(Fa \wedge \neg Gy)$	......8,∃I
10.	$\forall xFx \wedge \neg\forall xGx, \neg Gb$	$\vdash$	$\forall x\exists y(Fx \wedge \neg Gy)$	......9,∀I
11.	$\forall xFx \wedge \neg\forall xGx$	$\vdash$	$\forall x\exists y(Fx \wedge \neg Gy)$	..4,10,∃E

7. Here is something obvious: if there are at least two Fs, there is at least one F. Let's make sure that our way of counting delivers this result. Prove $\exists x[Fx \wedge \exists y(Gy \wedge x{\neq}y)] \vdash \exists x Fx$. Hint: for how to do the first few lines, consider the next problem.

1.	$\exists x[Fx \wedge \exists y(Gy \wedge x{\neq}y)]$	$\vdash \exists x[Fx \wedge \exists y(Gy \wedge x{\neq}y)]$	. . A
2.	$Fa \wedge \exists y(Gy \wedge a{\neq}y)$	$\vdash Fa \wedge \exists y(Gy \wedge a{\neq}y)$	. . . . A
3.	$Fa \wedge \exists y(Gy \wedge a{\neq}y)$	$\vdash Fa$	. . . . 2,∧E
4.	$Fa \wedge \exists y(Gy \wedge a{\neq}y)$	$\vdash \exists x Fx$	. . . . 3,∃I
5.	$\exists x[Fx \wedge \exists y(Gy \wedge x{\neq}y)]$	$\vdash \exists x Fx$	. . . . 1,4,∃E

8. Prove $\exists x(Fx \wedge \exists y Gy) \vdash \exists x \exists y(Fx \wedge Gy)$. Here are the first few lines.

1.	$\exists x(Fx \wedge \exists y Gy)$	$\vdash \exists x(Fx \wedge \exists y Gy)$	. . . . A
2.	$Fa \wedge \exists y Gy$	$\vdash Fa \wedge \exists y Gy$	. . . . A
3.	$Fa \wedge \exists y Gy$	$\vdash Fa$	. . . . 2,∧E
4.	$Fa \wedge \exists y Gy$	$\vdash \exists y Gy$	. . . . 2,∧E

5.	Gb	$\vdash Gb$	. . . . A
6.	$Fa \wedge \exists y Gy, Gb$	$\vdash Fa \wedge Gb$	. . . . 3,5,∧I
7.	$Fa \wedge \exists y Gy, Gb$	$\vdash \exists y(Fa \wedge Gy)$	. . . . 6,∃I
8.	$Fa \wedge \exists y Gy, Gb$	$\vdash \exists x \exists y(Fx \wedge Gy)$	. . . . 7,∃I
9.	$Fa \wedge \exists y Gy$	$\vdash \exists x \exists y(Fx \wedge Gy)$	. . . . 4,8,∃E
10.	$\exists x(Fx \wedge \exists y Gy)$	$\vdash \exists x \exists y(Fx \wedge Gy)$	. . . . 1,9,∃E

9. In Section 5.14 I claim without explanation that formulas 4 and 5 say the same thing. If I am right, we expect a derivation from formula 4 to formula 5. Here is a such a derivation. Add missing items (note: $\neg(a{\neq}b)$ is the same as $\neg\neg(a{=}b)$):

   To aid visibility, let

   Γ: $\exists x[Fx \wedge \neg\exists y(Fy \wedge x{\neq}y)]$

   Δ: $Fa \wedge \neg\exists y(Fy \wedge a{\neq}y)]$

   Notice Δ has a constant so we need to be mindful of that.

1.	Γ	$\vdash \exists x[Fx \wedge \neg\exists y(Fy \wedge x{\neq}y)]$	A
2.	Δ	$\vdash Fa \wedge \neg\exists y(Fy \wedge a{\neq}y)$	A
3.	Δ	$\vdash \underline{\neg\exists y(Fy \wedge a{\neq}y)}$	2, $\wedge$E
4.		$\vdash \underline{\neg\exists y(Fy \wedge a{\neq}y) \supset \forall y\neg(Fy \wedge a{\neq}y)}$	QE
5.	Δ	$\vdash \forall y\neg(Fy \wedge a{\neq}y)$	3,4,$\supset$E
6.	Δ	$\vdash \underline{\neg(Fb \wedge a{\neq}b)}$	5,$\forall$E
7.		$\vdash \neg(Fb \wedge a{\neq}b) \supset [\neg Fb \vee \neg(a{\neq}b)]$	DM
8.	Δ	$\vdash \underline{\neg Fb \vee \neg(a{\neq}b)}$	6,7,$\supset$E
9.		$\vdash [\neg Fb \vee \neg(a{\neq}b)] \supset [\neg\neg Fb \supset \neg(a{\neq}b)]$	EL
10.	$\underline{\Delta}$	$\vdash \underline{\neg\neg Fb \supset \neg(a{\neq}b)}$	8, 9, $\supset$E
11.	Fb	$\vdash Fb$	A
12.		$\vdash Fb \supset \neg\neg Fb$	DN
13.	Fb	$\vdash \neg\neg Fb$	$\underline{\text{11,12,}\supset\text{E}}$
14.	Δ, Fb	$\vdash \neg(a{\neq}b)$	10,13,$\supset$E
15.	Δ, Fb	$\vdash a{=}b$	$\underline{\text{14,}\neg\text{E}}$
16.	$\underline{\Delta}$	$\vdash \underline{Fb \supset a{=}b}$	15,$\supset$I
17.	Δ	$\vdash \underline{\forall y(Fy \supset a{=}y)}$	16,$\forall$I
18.	Δ	$\vdash \underline{Fa}$	$\underline{\text{2,}\wedge\text{E}}$
19.	Δ	$\vdash Fa \wedge \forall y(Fy \supset a{=}y)$	17,18,$\wedge$I
20.	$\underline{\Delta}$	$\vdash \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$	19,$\exists$I
21.	Γ	$\vdash \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$	$\underline{\text{1,20,}\exists\text{E}}$

10. Here is another way of saying that there is exactly one thing:

$\exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$

Prove $\exists x[Fx \wedge \forall y(Fy \supset x{=}y)] \vdash \exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$. Here are the first two lines:

1. $\exists x[Fx \wedge \forall y(Fy \supset x{=}y)] \quad \vdash \exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$. A
2. $Fa \wedge \forall y(Fy \supset a{=}y) \quad \vdash Fa \wedge \forall y(Fy \supset a{=}y)$ ......A

3.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash Fa$	......................2,∧E
4.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash \forall y(Fy \supset a{=}y)$	...........2,∧E
5.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash Fb \supset a{=}b$	................4,∀E
6.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash Fa \wedge (Fb \supset a{=}b)$	.... 3,5,∧I
7.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash \forall y[Fa \wedge (Fy \supset a{=}y)]$	....6,∀I
8.	$Fa \wedge \forall y(Fy \supset a{=}y)$	$\vdash \exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$	. 7,∃I
9.	$\exists x[Fx \wedge \forall y(Fy \supset x{=}y)]$	$\vdash \exists x \forall y[Fx \wedge (Fy \supset x{=}y)]$	1,8,∃E

11. Prove $\exists x \forall y(Fx \wedge Gy) \vdash \exists x(Fx \wedge \forall y Gy)$. (This is a simplified version of the reverse of the above).

1.	$\exists x \forall y(Fx \wedge Gy)$	$\vdash \exists x \forall y(Fx \wedge Gy)$	..............A
2.	$\forall y(Fa \wedge Gy)$	$\vdash \forall y(Fa \wedge Gy)$	.................A
3.	$\forall y(Fa \wedge Gy)$	$\vdash Fa \wedge Gb$	...................2,∀E
4.	$\forall y(Fa \wedge Gy)$	$\vdash Gb$	........................3,∧E
5.	$\forall y(Fa \wedge Gy)$	$\vdash \forall y Gy$	.... 4,∀I
6.	$\forall y(Fa \wedge Gy)$	$\vdash Fa$	.... 3,∧E
7.	$\forall y(Fa \wedge Gy)$	$\vdash Fa \wedge \forall y Gy$	.... 5,6,∧I
8.	$\forall y(Fa \wedge Gy)$	$\vdash \exists x(Fx \wedge \forall y Gy)$	.... 7,∃I
9.	$\exists x \forall y(Fx \wedge Gy)$	$\vdash \exists x(Fx \wedge \forall y Gy)$	.... 1,8,∃E