Exercises for 5.5–5.6

1. Consider the following formula:

   $\forall x\big[(Px \vee Sx) \supset Cx\big]$

   (a) Construct an interpretation that makes the above formula true.

   (b) Construct an interpretation that makes the above formula false.

> **Answer Key**
>
> The following models are just examples.
>
> (a) An intuitive interpretation: Let Px mean that x is a Pomona student, Sx mean that x is a Scripps student, and Cx mean that x is a 5C student. On this interpretation, the above formula is true.
>
> A formal model:
>
> - Domain of discourse: Charlie, Logan, Rowan, Sidney.
> - Referent of constants: c refers to Charlie, l refers to Logan, r refers to Rowan, s refers to Sidney.
> - Extension of P: Charlie and Logan. Extension of S: Rowan and Sidney. Extension of C: Charlie, Logan, Rowan, and Sidney.
>
> (b) An intuitive interpretation: Let Px mean that x is a Pitzer student, Sx mean that x is a Scripps student, and Cx mean that x is a CMC student. On this interpretation, the above formula is false.
>
> A formal model:
>
> - Domain of discourse: Charlie, Logan, Rowan, Sidney.
> - Referent of constants: c refers to Charlie, l refers to Logan, r refers to Rowan, s refers to Sidney.
> - Extension of P: Charlie and Logan. Extension of S: Rowan and Sidney. Extension of C: empty.

2. Consider the following formula:

   $Pa \supset \forall x(Rx \supset \neg Pa)$

   (a) Construct an interpretation that makes the formula true.

   (b) Construct an interpretation that makes the formula false.

Answer Key

(a) Notice that the whole formula is a conditional so that if Pa is false, the whole is true. We therefore only need to make sure Pa is false in our model.

- Domain of discourse: Ben, Jerry.
- Referents of constants: a refers to Ben, b refers to Jerry.
- Extension of P: empty. Extension of R: Ben and Jerry.

(b) Pa must be true, and $\forall x(Rx \supset \neg Pa)$ has to be false. The latter requires that $R\kappa \supset \neg Pa$ be false for some constant κ.

- Domain of discourse: Ben, Jerry.
- Referents of constants: a refers to Ben, b refers to Jerry.
- Extension of P: Ben. Extension of R: Ben and Jerry.

3. Consider the following formula:

$\forall x \forall y(Rxy \supset Ryx)$

(a) Construct an interpretation that makes the formula true.

(b) Construct an interpretation that makes the formula false.

Answer Key

(a) Intuitively, any interpretation of R as a symmetric relationship makes the formula true.

- Domain of discourse: Jill, Joe, George, Laura.
- Referents of constants: j_1 refers to Jill, j_2 refers to Joe, g refers to George, l refers to Laura.
- Extension of R: the ordered pairs <Jill, Joe>, <Joe,Jill>, <George, Laura>, <Laura, George>.

(Think of Rxy as meaning x is married to y.)

(b) Intuitively, any interpretation of R as an asymmetric relationship makes the formula false.

- Domain of discourse: Kamala, Mike.
- Referents of constants: k refers to Kamala, m refers to Mike.

- Extension of R: the ordered pair <Kamala, Mike>.

(Think of Rxy as meaning x is the successor of y as vice president.)

4. Consider the following formula:

   $\exists xFx \supset \neg\forall x\neg Fx$

   Explain why it is not possible to construct an interpretation that makes this formula false.

   Answer Key

   Since the formula is a conditional, its being false require the truth of $\exists xFx$ and the falsity of $\neg\forall x\neg Fx$. If $\exists xFx$ is true, then the extension of F has at least one member. On the other hand, if $\neg\forall x\neg Fx$ is false, then $\forall x\neg Fx$ is true which means that the extension of F is empty. Since the extension of F cannot be both empty and non-empty, there can be no model that makes the above formula false.

5. Consider:

   $\exists xFx \wedge \forall x\neg Fx$

   Explain why it is not possible to construct an interpretation that makes this formula true.

   Answer Key

   Since the sentence is a conjunction, its truth requires the truth of $\exists xFx$ and the truth of $\forall x\neg Fx$. But no model can make both true (see above for a bit more detail).

6. Which of the following are logical truths? For those that are not, provide counterexamples. (A counterexample is a quick of providing an interpretation that makes the sentence false. For instance, consider $\forall x(Mx \wedge Hx) \supset (\forall xMx \vee \forall xHx)$. Here is a counterexample: suppose everyone in class is a CMC or a HMC student. It doesn't follow that everyone is a CMC student or everyone is a HMC student—maybe there's a mix of the two groups).

(a) $\forall x(Fx \supset Gx) \supset (\forall xFx \supset \forall xGx)$

(b) $\exists x(Fx \supset Gx) \supset (\exists xFx \supset \exists xGx)$

(c) $(\exists xFx \land \exists xGx) \supset \exists x(Fx \land Gx)$

(d) $\exists x(Fx \land Gx) \supset (\exists xFx \land \exists xGx)$

Answer Key

(a) and (d) are logical truths.

(b), (c) are not logical truths. You can see that by constructing counterexamples. E.g.: (b) It might that there is someone who gets grumpy when hungry, but that does not mean that if there is someone hungry, there is someone grumpy (the hungry person need not be one who gets grumpy when hungry).

(c) There are vegans, and there are carnivores. It does not follow that there are some who are both vegan and carnivore.