Extra Exercises for Derivations/Proofs

1. For each of the following short derivations, explain why they are illegal. (1pt each)

   (a)
   1. $p \quad \vdash q$ . . . . premise
   2. $p \quad \vdash q \wedge r$ . . . . 1,$\wedge$I

   . . . .

   . . . .

   . . . .

   (b)
   1. $p \wedge p \quad \vdash r$ . . . . premise
   2. $p \quad \vdash r$ . . . . 1,$\wedge$E

   . . . .

   . . . .

   . . . .

   (c)
   1. $p \wedge p \quad \vdash r$ . . . . premise
   2. $p \quad \vdash r$ 1

   . . . .

   . . . .

   . . . .

   (d)
   1. $\Gamma \quad \vdash s \vee t$ . . . . premise
   2. $\Gamma \quad \vdash s$ . . . . 1,$\vee$E

   . . . .

   . . . .

   . . . .

   (e)
   1. $s \quad \vdash p \supset q$ . . . . premise
   2. $s \vee t \quad \vdash p \supset q$ . . . . 1,$\vee$I

   . . . .

   . . . .

   . . . .

(f)

1. $\Gamma, p \vdash q$ .... premise
2. $\Delta, p \vdash r$ .... premise
3. $\Gamma, \Delta \vdash \neg p$ .... 1,2,¬I

....

....

....

(g)

1. $\Gamma, p \vdash q$ .... premise
2. $\Delta, \neg p \vdash q$ .... premise
3. $\Gamma, \Delta \vdash \neg q$ .... 1,2,¬I

....

....

....

(h)

1. $\Delta \vdash p \supset q$ .... premise
2. $\Delta, p \vdash q$ .... 1,⊃E

....

....

....

(i)

1. $\Delta \vdash p \supset q$ .... premise
2. $\Gamma \vdash q$ .... premise
3. $\Delta, \Gamma \vdash p$ .... ⊃E

....

....

....

(j)

1. $\Delta, s \vdash t$ .... premise
2. $\Delta \vdash t \supset s$ .... 1,⊃I

....

....

....

2. Here is a proof of Excluded Middle that's different from the one given in the readings. Add any missing datums:

1.	___	$\vdash p \wedge \neg p$	A
2.	___	$\vdash p$	1,∧E
3.	___	$\vdash \neg p$	1,∧E
4.	___	$\vdash \neg(p \wedge \neg p)$	2,3,¬I
5.	___	$\vdash \neg(p \vee \neg p)$	A
6.	___	$\vdash p$	A
7.	___	$\vdash p \vee \neg p$	6,∨I
8.	___	$\vdash \neg(p \vee \neg p)$	5
9.	___	$\vdash \neg p$	7,8,¬I
10.	___	$\vdash \neg p$	A
11.	___	$\vdash p \vee \neg p$	10,∨I
12.	___	$\vdash \neg(p \vee \neg p)$	5
13.	___	$\vdash \neg\neg p$	11,12,¬I
14.	___	$\vdash p$	13,¬E
15.	___	$\vdash p \wedge \neg p$	9,14,∧I
16.	___	$\vdash \neg(p \wedge \neg p)$	4
17.	___	$\vdash \neg\neg(p \vee \neg p)$	15,16,¬I
18.	___	$\vdash p \vee \neg p$	17,¬E

3. Add missing annotations:

1.	Γ	$\vdash P \vee Q$	premise
2.	Δ	$\vdash P \supset R$	premise
3.	Θ	$\vdash Q \supset S$	premise
4.	P	$\vdash P$	__
5.	Δ, P	$\vdash R$	__
6.	Δ, P	$\vdash R \vee S$	__
7.	Q	$\vdash Q$	__
8.	Θ, Q	$\vdash S$	__
9.	Θ, Q	$\vdash R \vee S$	__
10.	Γ, Δ, Θ	$\vdash R \vee S$	__

4. Fill in the missing items.

1.	Γ	$\vdash P \vee (Q \vee R)$	premise
2.	P	$\vdash P$	A
3.	__	$\vdash$ __	A
4.	R	$\vdash (P \vee Q) \vee R$	3,__
5.	__	$\vdash$ __	2,∨I
6.	__	$\vdash$ __	A
7.	Q	$\vdash$ __	6,__
8.	__	$\vdash$ __	5,∨I
9.	$Q \vee R$	$\vdash Q \vee R$	A
10.	__	$\vdash (P \vee Q) \vee R$	7,∨I
11.	__	$\vdash$ __	__
12.	Γ	$\vdash (P \vee Q) \vee R$	1,8,11,∨E

5. The following contains one illegal move. Where is it?

1.	Γ	$\vdash W \vee S$	premise
2.	Δ	$\vdash \neg W$	premise
3.	S	$\vdash S$	A
4.	$S, \neg W$	$\vdash S$	3
5.	S	$\vdash \neg W \supset S$	4,⊃I
6.	Δ, S	$\vdash S$	2,5,⊃E
7.	W	$\vdash W$	A
8.	$\neg W$	$\vdash \neg W$	A
9.	$W, \neg S$	$\vdash W$	7
10.	$\neg W, \neg S$	$\vdash \neg W$	8
11.	$W, \neg W$	$\vdash \neg\neg S$	9,10,¬I
12.	W	$\vdash \neg W \supset \neg\neg S$	11,⊃I
13.	W	$\vdash \neg W \supset S$	12,¬E
14.	Δ, W	$\vdash S$	2,13,⊃E
15.	Γ, Δ	$\vdash S$	1,6,14,∨E

6. Fill in the missing items.

1.	Γ	$\vdash W \vee (S \supset \neg T)$	premise
2.	Δ	$\vdash \neg W$	premise
3.	__	$\vdash$ __	A
4.	__	$\vdash$ __	2
5.	__	$\vdash$ __	3
6.	__	$\vdash$ __	4,5,¬I
7.	__	$\vdash$ __	6,¬E
8.	__	$\vdash$ __	A
9.	Γ, Δ	$\vdash S \supset \neg T$	1,7,8,∨E

7. Here is yet another proof of Excluded Middle. Fill in any missing items.

1.	$\neg(p \vee \neg p)$	$\vdash \neg(p \vee \neg p)$	A
2.	$\neg p$	$\vdash \neg p$	A
3.	__	$\vdash$ __	2,$\vee$I
4.	__	$\vdash$ __	1
5.	__	$\vdash$ __	3,4,$\neg$I
6.	p	$\vdash p$	A
7.	__	$\vdash$ __	6,$\vee$I
8.	__	$\vdash$ __	1
9.	__	$\vdash$ __	7,8,$\neg$I
10.	__	$\vdash$ __	5,9,$\neg$I
11.		$\vdash p \vee \neg p$	10,$\neg$E

8. Prove the following sequents (keep in mind what it means to prove a sequent). Each of these require only three lines:

   (a) $p, q \vdash p \vee q$

   (b) $\neg p, q \vdash p \vee q$

   (c) $p, \neg q \vdash p \vee q$

9. Fill in the missing items of the following proof of $\neg p, \neg q \vdash \neg(p \vee q)$:

1.	$\neg p$	$\vdash \neg p$	A
2.	$\neg q$	$\vdash \neg q$	A
3.	$p \vee q$	$\vdash p \vee q$	A
4.	p	$\vdash p$	A
5.	__	$\vdash$ __	1
6.	__	$\vdash$ __	4
7.	$\neg p, p$	$\vdash \neg(p \vee q)$	5,6,¬I
8.	q	$\vdash q$	A
9.	__	$\vdash$ __	8
10.	__	$\vdash$ __	2
11.	__	$\vdash \neg(p \vee q)$	9,10,¬I
12.	$p \vee q, \neg p, \neg q$	$\vdash \neg(p \vee q)$	__
13.	$\neg p, \neg q$	$\vdash \neg(p \vee q)$	__

10. Prove Implication: $\vdash (p \supset q) \supset (\neg p \vee q)$. Hint: you can adapt one of the derivations we have done earlier in our exercises.

11. Prove Contraposition. Hint: you can adapt one of the derivations in the earlier exercises.

12. Derive from $\Gamma \vdash \neg(\neg P \vee \neg Q)$ to $\Gamma \vdash P \wedge Q$. Hint: you can adapt one of our earlier exercises. ....

13. Prove $\vdash \neg(p \wedge q) \supset (\neg p \vee \neg q)$. Hint: the overall proof is a proof by contradiction (assume the negation of the consequent of the conditional), and the previous question shows how to do most of the needed work.

14. Construct a derivation from $\Gamma \vdash P \vee Q$ and $\Delta \vdash Q \supset R$ to $\Gamma, \Delta \vdash P \vee R$. (Here is an example of this in plain English: Joe is eating Chinese place or Italian. If he is eating Italian, he is eating eggplant parmigiana. So Joe is eating Chinese or he is eating eggplant parmigiana.) Hint: $\vee$E is your friend.

15. Prove Double Negation Introduction. Hint: use $\neg$I.

16. In a single proof, prove $p, q \vdash p \supset q$ and $\neg p, q \vdash p \supset q$ (recall that a proof proves each sequent; apart from Assumption Introduction, you only need $\supset$I and sequent rewrites).

17. Prove $p, \neg q \vdash \neg(p \supset q)$. Hint: assume $p \supset q$, p, and $\neg q$.

18. Prove $\neg p, \neg q \vdash p \supset q$. Hint: adapt the derivation from $\Gamma \vdash P$ to $\Gamma \vdash \neg P \supset Q$.

19. Derive from $p, q \vdash r$ to $\vdash (p \wedge q) \supset r$.