Extra Exercises for Derivations/Proofs

1. For each of the following short derivations, explain why they are illegal. (1pt each)

   (a)

1.	p	$\vdash q$	premise
2.	p	$\vdash q \wedge r$	1,$\wedge$I

   You need to refer to two lines to use $\wedge$I.

   (b)

1.	$p \wedge p$	$\vdash r$	premise
2.	p	$\vdash r$	1,$\wedge$E

   $\wedge$E cannot change datum.

   (c)

1.	$p \wedge p$	$\vdash r$	premise
2.	p	$\vdash r$	1

   Sequent rewrite cannot eliminate a conjunct in the datum.

   (d)

1.	Γ	$\vdash s \vee t$	premise
2.	Γ	$\vdash s$	1,$\vee$E

   $\vee$E does not work that way.

   (e)

1.	s	$\vdash p \supset q$	premise
2.	$s \vee t$	$\vdash p \supset q$	1,$\vee$I

   Cannot use $\vee$I to change datum.

(f)

1.	Γ, p	$\vdash q$	premise
2.	Δ, p	$\vdash r$	premise
3.	Γ, Δ	$\vdash \neg p$	1,2,$\neg$I

> Lines 1 and 2 do not contradicting succedents.

(g)

1.	Γ, p	$\vdash q$	premise
2.	$\Delta, \neg p$	$\vdash q$	premise
3.	Γ, Δ	$\vdash \neg q$	1,2,$\neg$I

> $\neg$I misapplied. Contradiction must be on the succedent side.

(h)

1.	Δ	$\vdash p \supset q$	premise
2.	Δ, p	$\vdash q$	1,$\supset$E

> Need another line to use $\supset$E.

(i)

1.	Δ	$\vdash p \supset q$	premise
2.	Γ	$\vdash q$	premise
3.	Δ, Γ	$\vdash p$	$\supset$E

> $\supset$E misapplied. You need the antecedent of the conditional in the succedent of second sequent.

(j)

1.	Δ, s	$\vdash t$	premise
2.	Δ	$\vdash t \supset s$	1,$\supset$I

> $\supset$I misapplied. The item from the datum goes into the antecedent of the conditional.

2. Here is a proof of Excluded Middle that's different from the one given in the readings. Add any missing datums:

1.	$\underline{p \wedge \neg p}$	$\vdash p \wedge \neg p$	A
2.	$\underline{p \wedge \neg p}$	$\vdash p$	1,$\wedge$E

3.	$\underline{p \wedge \neg p}$	$\vdash \neg p$	1,∧E
4.		$\vdash \neg(p \wedge \neg p)$	2,3,¬I
5.	$\underline{\neg(p \vee \neg p)}$	$\vdash \neg(p \vee \neg p)$	A
6.	$\underline{p}$	$\vdash p$	A
7.	$\underline{p}$	$\vdash p \vee \neg p$	6,∨I
8.	$\underline{\neg(p \vee \neg p), p}$	$\vdash \neg(p \vee \neg p)$	5
9.	$\underline{\neg(p \vee \neg p)}$	$\vdash \neg p$	7,8,¬I
10.	$\underline{\neg p}$	$\vdash \neg p$	A
11.	$\underline{\neg p}$	$\vdash p \vee \neg p$	10,∨I
12.	$\underline{\neg(p \vee \neg p), \neg p}$	$\vdash \neg(p \vee \neg p)$	5
13.	$\underline{\neg(p \vee \neg p)}$	$\vdash \neg\neg p$	11,12,¬I
14.	$\underline{\neg(p \vee \neg p)}$	$\vdash p$	13,¬E
15.	$\underline{\neg(p \vee \neg p)}$	$\vdash p \wedge \neg p$	9,14,∧I
16.	$\underline{\neg(p \vee \neg p)}$	$\vdash \neg(p \wedge \neg p)$	4
17.		$\vdash \neg\neg(p \vee \neg p)$	15,16,¬I
18.		$\vdash p \vee \neg p$	17,¬E

3. Add missing annotations:

1.	Γ	$\vdash P \vee Q$	premise
2.	Δ	$\vdash P \supset R$	premise
3.	Θ	$\vdash Q \supset S$	premise
4.	P	$\vdash P$	$\underline{\text{A}}$
5.	Δ, P	$\vdash R$	$\underline{\text{2,4,}\supset\text{E}}$
6.	Δ, P	$\vdash R \vee S$	$\underline{\text{5,}\vee\text{I}}$
7.	Q	$\vdash Q$	$\underline{\text{A}}$
8.	Θ, Q	$\vdash S$	$\underline{\text{3,7,}\supset\text{E}}$
9.	Θ, Q	$\vdash R \vee S$	$\underline{\text{8,}\vee\text{I}}$
10.	Γ, Δ, Θ	$\vdash R \vee S$	$\underline{\text{1,6,9,}\vee\text{E}}$

4. Fill in the missing items.

1.	Γ	$\vdash P \vee (Q \vee R)$	premise
2.	P	$\vdash P$	A
3.	$\underline{R}$	$\vdash \underline{R}$	A
4.	R	$\vdash (P \vee Q) \vee R$	3,$\underline{\vee\text{I}}$
5.	$\underline{P}$	$\vdash \underline{P \vee Q}$	2,$\vee$I
6.	$\underline{Q}$	$\vdash \underline{Q}$	A
7.	Q	$\vdash \underline{P \vee Q}$	6,$\underline{\vee\text{I}}$
8.	$\underline{P}$	$\vdash \underline{(P \vee Q) \vee R}$	5,$\vee$I
9.	$Q \vee R$	$\vdash Q \vee R$	A
10.	$\underline{Q}$	$\vdash (P \vee Q) \vee R$	7,$\vee$I
11.	$\underline{Q \vee R}$	$\vdash \underline{(P \vee Q) \vee R}$	$\underline{\text{4,9,10,}\vee\text{E}}$
12.	Γ	$\vdash (P \vee Q) \vee R$	1,8,11,$\vee$E

5. The following contains one illegal move. Where is it?

1.	Γ	$\vdash W \vee S$	premise
2.	Δ	$\vdash \neg W$	premise
3.	S	$\vdash S$	A
4.	$S, \neg W$	$\vdash S$	3
5.	S	$\vdash \neg W \supset S$	4,⊃I
6.	Δ, S	$\vdash S$	2,5,⊃E
7.	W	$\vdash W$	A
8.	$\neg W$	$\vdash \neg W$	A
9.	$W, \neg S$	$\vdash W$	7
10.	$\neg W, \neg S$	$\vdash \neg W$	8
11.	$W, \neg W$	$\vdash \neg\neg S$	9,10,¬I
12.	W	$\vdash \neg W \supset \neg\neg S$	11,⊃I
13.	W	$\vdash \neg W \supset S$	12,¬E
14.	Δ, W	$\vdash S$	2,13,⊃E
15.	Γ, Δ	$\vdash S$	1,6,14,∨E

Answer Key

The move to line 13 is not allowed by our inference rules. Notice that the main connective of the succedent of line 12 is the conditional.

6. Fill in the missing items.

1.	Γ	$\vdash W \vee (S \supset \neg T)$	premise
2.	Δ	$\vdash \neg W$	premise
3.	$\underline{W}$	$\vdash \underline{W}$	A
4.	$\underline{\Delta, \neg(S \supset \neg T)}$	$\vdash \underline{\neg W}$	2
5.	$\underline{W, \neg(S \supset \neg T)}$	$\vdash \underline{W}$	3
6.	$\underline{\Delta, W}$	$\vdash \underline{\neg\neg(S \supset \neg T)}$	4,5,¬I
7.	$\underline{\Delta, W}$	$\vdash \underline{(S \supset \neg T)}$	6,¬E
8.	$\underline{(S \supset \neg T)}$	$\vdash \underline{(S \supset \neg T)}$	A
9.	Γ, Δ	$\vdash S \supset \neg T$	1,7,8,∨E

7. Here is yet another proof of Excluded Middle. Fill in any missing items.

1.	$\neg(p \vee \neg p)$	$\vdash \neg(p \vee \neg p)$	A
2.	$\neg p$	$\vdash \neg p$	A
3.	$\underline{\neg p}$	$\vdash \underline{p \vee \neg p}$	2,$\vee$I
4.	$\underline{\neg(p \vee \neg p), \neg p}$	$\vdash \underline{\neg(p \vee \neg p)}$	1
5.	$\underline{\neg(p \vee \neg p)}$	$\vdash \underline{\neg\neg p}$	3,4,$\neg$I
6.	p	$\vdash p$	A
7.	$\underline{p}$	$\vdash \underline{p \vee \neg p}$	6,$\vee$I
8.	$\underline{\neg(p \vee \neg p), p}$	$\vdash \underline{\neg(p \vee \neg p)}$	1
9.	$\underline{\neg(p \vee \neg p)}$	$\vdash \underline{\neg p}$	7,8,$\neg$I
10.		$\vdash \underline{\neg\neg(p \vee \neg p)}$	5,9,$\neg$I
11.		$\vdash p \vee \neg p$	10,$\neg$E

8. Prove the following sequents (keep in mind what it means to prove a sequent). Each of these require only three lines:

(a) $p, q \vdash p \vee q$

1.	p	$\vdash p$	A
2.	p	$\vdash p \vee q$	1,$\vee$I
3.	p, q	$\vdash p \vee q$	2

(b) $\neg p, q \vdash p \vee q$

1.	q	$\vdash q$	A
2.	q	$\vdash p \vee q$	1,$\vee$I
3.	$\neg p, q$	$\vdash p \vee q$	2

(c) $p, \neg q \vdash p \vee q$

1.	p	$\vdash p$	............................A
2.	p	$\vdash p \lor q$	.... 1,$\lor$I
3.	$p, \neg q$	$\vdash p \lor q$	2

9. Fill in the missing items of the following proof of $\neg p, \neg q \vdash \neg(p \vee q)$:

1.	$\neg p$	$\vdash \neg p$	A
2.	$\neg q$	$\vdash \neg q$	A
3.	$p \vee q$	$\vdash p \vee q$	A
4.	p	$\vdash p$	A
5.	$\underline{\neg p, p \vee q}$	$\vdash \underline{\neg p}$	1
6.	$\underline{p, p \vee q}$	$\vdash \underline{p}$	4
7.	$\neg p, p$	$\vdash \neg(p \vee q)$	5,6,¬I
8.	q	$\vdash q$	A
9.	$\underline{q, p \vee q}$	$\vdash \underline{q}$	8
10.	$\underline{\neg q, p \vee q}$	$\vdash \underline{\neg q}$	2
11.	$\underline{\neg q, q}$	$\vdash \neg(p \vee q)$	9,10,¬I
12.	$p \vee q, \neg p, \neg q$	$\vdash \neg(p \vee q)$	<u>3,7,11, ∨E</u>
13.	$\neg p, \neg q$	$\vdash \neg(p \vee q)$	<u>3,12,¬I</u>

10. Prove Implication: $\vdash (p \supset q) \supset (\neg p \vee q)$. Hint: you can adapt one of the derivations we have done earlier in our exercises.

Answer Key

You can adapt a derivation we have done earlier in our exercises.

1.	$p \supset q$	$\vdash p \supset q$	A
2.	$\neg(\neg p \vee q)$	$\vdash \neg(\neg p \vee q)$	A
3.	$\neg p$	$\vdash \neg p$	A
4.	$\neg p$	$\vdash \neg p \vee q$	3,∨I
5.	$\neg(\neg p \vee q), \neg p$	$\vdash \neg(\neg p \vee q)$	2
6.	$\neg(\neg p \vee q)$	$\vdash \neg\neg p$	4,5,¬I
7.	$\neg(\neg p \vee q)$	$\vdash p$	6,¬E
8.	$p \supset q, \neg(\neg p \vee q)$	$\vdash q$	1,7,⊃E
9.	$p \supset q, \neg(\neg p \vee q)$	$\vdash \neg p \vee q$	8,∨I
10.	$p \supset q$	$\vdash \neg\neg(\neg p \vee q)$	2,9,¬I
11.	$p \supset q$	$\vdash \neg p \vee q$	10,¬E
12.		$\vdash (p \supset q) \supset (\neg p \vee q)$	12,⊃I

11. Prove Contraposition. Hint: you can adapt one of the derivations in the earlier exercises.

Answer Key

You can adapt a derivation in one of our earlier exercises.

1.	$p \supset q$	$\vdash p \supset q$	A
2.	$\neg q$	$\vdash \neg q$	A
3.	p	$\vdash p$	A
4.	$p \supset q, p$	$\vdash q$	1,3,$\supset$E
5.	$\neg q, p$	$\vdash \neg q$	2
6.	$p \supset q, \neg q$	$\vdash \neg p$	4,5,$\neg$I
7.	$p \supset q$	$\vdash \neg q \supset \neg p$	6,$\supset$I
8.		$\vdash (p \supset q) \supset (\neg q \supset \neg p)$	7,$\supset$I

12. Derive from $\Gamma \vdash \neg(\neg P \vee \neg Q)$ to $\Gamma \vdash P \wedge Q$. Hint: you can adapt one of our earlier exercises.

Answer Key

1.	Γ	$\vdash \neg(\neg P \vee \neg Q)$	premise
2.	$\neg P$	$\vdash \neg P$	A
3.	$\neg P$	$\vdash \neg P \vee \neg Q$	2,∨I
4.	$\Gamma, \neg P$	$\vdash \neg(\neg P \vee \neg Q)$	1
5.	Γ	$\vdash \neg\neg P$	3,4,¬I
6.	Γ	$\vdash P$	5,¬E
7.	$\neg Q$	$\vdash \neg Q$	A
8.	$\neg Q$	$\vdash \neg P \vee \neg Q$	7,∨I
9.	$\Gamma, \neg Q$	$\vdash \neg(\neg P \vee \neg Q)$	1
10.	Γ	$\vdash \neg\neg Q$	8,9,¬I
11.	Γ	$\vdash Q$	10,¬E
12.	Γ	$\vdash P \wedge Q$	6,11,∧I

13. Prove $\vdash \neg(p \wedge q) \supset (\neg p \vee \neg q)$. Hint: the overall proof is a proof by contradiction (assume the negation of the consequent of the conditional), and the previous question shows how to do most of the needed work.

Answer Key

Notice that lines 2 through 13 replicate the previous derivation where we plug $\neg(\neg p \vee \neg q)$ into Γ.

1.	$\neg(p \wedge q)$	$\vdash \neg(p \wedge q)$	.... A
2.	$\neg(\neg p \vee \neg q)$	$\vdash \neg(\neg p \vee \neg q)$	.... A
3.	$\neg p$	$\vdash \neg p$	.... A
4.	$\neg p$	$\vdash \neg p \vee \neg q$	.... 3,∨I
5.	$\neg(\neg p \vee \neg q), \neg p$	$\vdash \neg(\neg p \vee \neg q)$	 2
6.	$\neg(\neg p \vee \neg q)$	$\vdash \neg\neg p$	.... 4,5,¬I
7.	$\neg(\neg p \vee \neg q)$	$\vdash p$	.... 6,¬E
8.	$\neg q$	$\vdash \neg q$	.... A
9.	$\neg q$	$\vdash \neg p \vee \neg q$	.... 8,∨I
10.	$\neg(\neg p \vee \neg q), \neg q$	$\vdash \neg(\neg p \vee \neg q)$	 2
11.	$\neg(\neg p \vee \neg q)$	$\vdash \neg\neg q$	.... 9,10,¬I
12.	$\neg(\neg p \vee \neg q)$	$\vdash q$	.... 11,¬E
13.	$\neg(\neg p \vee \neg q)$	$\vdash p \wedge q$	.... 7,12,∧I
14.	$\neg(p \wedge q), \neg(\neg p \vee \neg q)$	$\vdash \neg(p \wedge q)$	 1
15.	$\neg(p \wedge q)$	$\vdash \neg\neg(\neg p \vee \neg q)$	...... 13,14,¬I
16.	$\neg(p \wedge q)$	$\vdash \neg p \vee \neg q$	.... 15,¬E

14. Construct a derivation from $\Gamma \vdash P \vee Q$ and $\Delta \vdash Q \supset R$ to $\Gamma, \Delta \vdash P \vee R$. (Here is an example of this in plain English: Joe is eating Chinese place or Italian. If he is eating Italian, he is eating eggplant parmigiana. So Joe is eating Chinese or he is eating eggplant parmigiana.) Hint: $\vee$E is your friend.

Answer Key

1.	Γ	$\vdash P \vee Q$	premise
2.	Δ	$\vdash Q \supset R$	premise
3.	P	$\vdash P$	A
4.	P	$\vdash P \vee R$	3,$\vee$I
5.	Q	$\vdash Q$	A
6.	Δ, Q	$\vdash R$	2,5,$\supset$E
7.	Δ, Q	$\vdash P \vee R$	6,$\vee$I
8.	Γ, Δ	$\vdash P \vee R$	1,4,7,$\vee$E

15. Prove Double Negation Introduction. Hint: use ¬I.

Answer Key

1.	p	$\vdash p$	A
2.	$\neg p$	$\vdash \neg p$	A
3.	$p, \neg p$	$\vdash p$	1
4.	p	$\vdash \neg\neg p$	2,3,¬I

16. In a single proof, prove $p, q \vdash p \supset q$ and $\neg p, q \vdash p \supset q$ (recall that a proof proves each sequent; apart from Assumption Introduction, you only need ⊃I and sequent rewrites).

Answer Key

1.	q	$\vdash q$	A
2.	q, p	$\vdash q$	1
3.	q	$\vdash p \supset q$	2,⊃I
4.	p, q	$\vdash p \supset q$	3
5.	$\neg p, q$	$\vdash p \supset q$	3

17. Prove $p, \neg q \vdash \neg(p \supset q)$. Hint: assume $p \supset q$, p, and $\neg q$.

Answer Key

1.	$p \supset q$	$\vdash p \supset q$	A
2.	p	$\vdash p$	A
3.	$\neg q$	$\vdash \neg q$	A
4.	$p \supset q, p$	$\vdash q$	1,2,⊃E
5.	$p \supset q, \neg q$	$\vdash \neg q$	3
6.	$p, \neg q$	$\vdash \neg(p \supset q)$	4,5,¬I

18. Prove $\neg p, \neg q \vdash p \supset q$. Hint: adapt the derivation from $\Gamma \vdash P$ to $\Gamma \vdash \neg P \supset Q$.

Answer Key

Notice that p is equivalent to $\neg(\neg p)$.

1.	$\neg p$	$\vdash \neg p$	A
2.	$\neg p, \neg q$	$\vdash \neg p$	1
3.	p	$\vdash p$	A
4.	$p, \neg q$	$\vdash p$	3
5.	$\neg p, p$	$\vdash \neg\neg q$	2,4,$\neg$I
6.	$\neg p, p$	$\vdash q$	5,$\neg$E
7.	$\neg p$	$\vdash p \supset q$	6,$\supset$I
8.	$\neg p, \neg q$	$\vdash p \supset q$	7

19. Derive from $p, q \vdash r$ to $\vdash (p \wedge q) \supset r$.

Answer Key

1.	p, q	$\vdash r$	premise
2.	$p \wedge q$	$\vdash p \wedge q$	A
3.	p	$\vdash q \supset r$	1,$\supset$I
4.	$p \wedge q$	$\vdash q$	2,$\wedge$E
5.	$p \wedge q, p$	$\vdash r$	3,4,$\supset$E
6.	$p \wedge q$	$\vdash p \supset r$	5,$\supset$I
7.	$p \wedge q$	$\vdash p$	2,$\wedge$E
8.	$p \wedge q$	$\vdash r$	6,7,$\supset$E
9.		$\vdash (p \wedge q) \supset r$	8,$\supset$I