1. For each of the following, indicate whether or not the inference is allowed. Yes means allowed. a, b are constants, and they do not appear in Γ and Δ.

(a)

1.	Fa	$\vdash Fa$	A
2.	Fa	$\vdash Gb \vee Fa$	1,∨I
3.	Fa	$\vdash \exists x Gx \vee Fa$	2,∃I

Yes/No

(b)

1.	Γ	$\vdash \exists x(Gx \wedge Fx)$	premise
2.	$Ga \wedge Fa$	$\vdash \exists x Gx$	premise
3.	Γ	$\vdash \exists x Gx$	1,2,∃E

Yes/No

(c)

1.	Γ	$\vdash \exists x(Gx \wedge Fx)$	premise
2.	$\Gamma, Ga \wedge Fa$	$\vdash \exists x Gx \wedge Fa$	premise
3.	Γ	$\vdash \exists x Gx \wedge Fa$	1,2,∃E

Yes/No

(d)

1.	Fa	$\vdash Fa$	A
2.	Fa	$\vdash Gb \vee Fa$	1,∨I
3.	Fa	$\vdash \exists x(Gx \vee Fa)$	2,∃I

Yes/No

(e)

1.	Fa	$\vdash Fa$	A
2.	Fa	$\vdash Gb \vee Fa$	2,∨I
3.	Fa	$\vdash \exists x Gx \vee Fa$	2,∃I

Yes/No

(f)

1.	$\forall x(Fx \supset Ga)$	$\vdash \forall x(Fx \supset Ga)$	A
2.	$\forall x(Fx \supset Ga)$	$\vdash Fa \supset Ga$	1,∀E
3.	$\forall x(Fx \supset Ga)$	$\vdash \forall x(Fx \supset Gx)$	2,∀I

Yes/No

(g)

1. $\Gamma \vdash \forall x Fx$ .... premise
2. $\Delta \vdash \forall y Gy$ .... premise
3. $\Gamma, \Delta \vdash \forall x \forall y (Fx \wedge Gy)$ .... 1,2,∧I

.... Yes/No

(h)

1. $\Gamma \vdash \forall x Fx \wedge Ga$ .... premise
2. $\Gamma \vdash \forall x \forall y (Fx \wedge Gy)$ .... 1,∀I

.... Yes/No

(i)

1. $\Gamma \vdash \forall x (Fx \wedge Ga)$ .... premise
2. $\Gamma \vdash \forall y \forall x (Fx \wedge Gy)$ .... 1,∀I

.... Yes/No

(j)

1. $\Gamma, \exists x \neg Fx \vdash \exists x Mx$ .... premise
2. $\Delta, \exists x \neg Fx \vdash \neg \exists x Mx$ .... premise
3. $\Gamma, \Delta \vdash \forall x Fx$ .... 1,2,¬I

.... Yes/No

2. Let's prove the theorems known as Quantifier Exchange. Here are the first four.

   For these four, do not appeal to Quantifier Exchange anywhere. Notice we stop one step short of the actual theorem. The theorems in questions are conditionals of the form $s_1 \supset s_2$. Instead of proving that, we stop at $s_1 \vdash s_2$ because the we can get to the conditional trivially by ⊃I. But the actual theorem is still the conditional!

   (a) $\exists x Fx \vdash \neg\forall x \neg Fx$. The proof of this is in the readings. See if you can reproduce it.

1.	$\exists x Fx$	$\vdash \exists x Fx$	A
2.	$\forall x \neg Fx$	$\vdash \forall x \neg Fx$	A
3.	Fa	$\vdash Fa$	A
4.	$\forall x \neg Fx$	$\vdash \neg Fa$	2,∀E
5.	$Fa, \forall x \neg Fx$	$\vdash Fa$	3
6.	Fa	$\vdash \neg\forall x \neg Fx$	4,5,¬I
7.	$\exists x Fx$	$\vdash \neg\forall x \neg Fx$	1,6,∃E

   (b) $\neg\forall x \neg Fx \vdash \exists x Fx$. Add the missing annotations (this is a little more efficient than the one in an earlier exercise).

1.	$\neg\forall x \neg Fx$	$\vdash \neg\forall x \neg Fx$	A
2.	$\neg\exists x Fx$	$\vdash \neg\exists x Fx$	A
3.	Fa	$\vdash Fa$	A
4.	$\underline{Fa}$	$\vdash \underline{\exists x Fx}$	<u>3,∃I</u>
5.		$\vdash Fa \supset \exists x Fx$	4,⊃I
6.		$\vdash \underline{(Fa \supset \exists x Fx) \supset (\neg\exists x Fx \supset \neg Fa)}$	<u>CP</u>
7.		$\vdash \underline{\neg\exists x Fx \supset \neg Fa}$	5,6,⊃E
8.	$\neg\exists x Fx$	$\vdash \neg Fa$	2,7,⊃E
9.	$\neg\exists x Fx$	$\vdash \forall x \neg Fx$	<u>8,∀I</u>
10.	$\underline{\neg\forall x \neg Fx, \neg\exists x Fx}$	$\vdash \underline{\neg\forall x \neg Fx}$	1
11.	$\underline{\neg\forall x \neg Fx}$	$\vdash \underline{\neg\neg\exists x Fx}$	9, 10, ¬I
12.	$\neg\forall x \neg Fx$	$\vdash \exists x Fx$	<u>11,¬E</u>

(c) $\forall xFx \vdash \neg\exists x\neg Fx$. Prove this sequent. Hint: assume the sentence in the datum, and assume the denial of the succedent of this sequent.

1.	$\forall xFx$	$\vdash$	$\forall xFx$	A
2.	$\exists x\neg Fx$	$\vdash$	$\exists x\neg Fx$	A
3.	$\neg Fa$	$\vdash$	$\neg Fa$	A
4.	$\forall xFx$	$\vdash$	Fa	1,∀E
5.	$\neg Fa, \exists x\neg Fx$	$\vdash$	$\neg Fa$	3
6.	$\forall xFx, \exists x\neg Fx$	$\vdash$	Fa	4
7.	$\forall xFx, \neg Fa$	$\vdash$	$\neg\exists x\neg Fx$	5,6,¬I
8.	$\forall xFx, \exists x\neg Fx$	$\vdash$	$\neg\exists x\neg Fx$	2,7,∃E
9.	$\forall xFx$	$\vdash$	$\neg\exists x\neg Fx$	2,8,¬I

(d) $\neg\exists x\neg Fx \vdash \forall xFx$. Prove this sequent. This was in an earlier exercise. See if you can reproduce it.

1.	$\neg\exists x\neg Fx$	$\vdash$	$\neg\exists x\neg Fx$	A
2.	$\neg Fa$	$\vdash$	$\neg Fa$	A
3.	$\neg Fa$	$\vdash$	$\exists x\neg Fx$	2,∃I
4.	$\neg\exists x\neg Fx, \neg Fa$	$\vdash$	$\neg\exists x\neg Fx$	1
5.	$\neg\exists x\neg Fx$	$\vdash$	$\neg\neg Fa$	3,4,¬I
6.	$\neg\exists x\neg Fx$	$\vdash$	Fa	5,¬E
7.	$\neg\exists x\neg Fx$	$\vdash$	$\forall xFx$	6,∀I

3. Let's prove the remaining theorems known as Quantifier Exchange. You may appeal to the theorems proven in the previous question. Hint: for the latter three, the first one below gives you something like a template.

(a) $\exists x \neg Fx \vdash \neg \forall x Fx$. Fill in the missing items.

1.	$\exists x \neg Fx$	$\vdash \exists x \neg Fx$	. . . . A
2.	$\forall x Fx$	$\vdash \forall x Fx$	. . . . A
3.		$\vdash \forall x Fx \supset \neg \exists x \neg Fx$	. . . . QE
4.	$\forall x Fx$	$\vdash \neg \exists x \neg Fx$	. . . . 2,3,⊃E
5.	$\exists x \neg Fx, \forall x Fx$	$\vdash \exists x \neg Fx$	 1
6.	$\exists x \neg Fx$	$\vdash \neg \forall x Fx$	. . . . 4,5,¬I

(b) $\neg \forall x Fx \vdash \exists x \neg Fx$. Prove this sequent.

1.	$\neg \forall x Fx$	$\vdash \neg \forall x Fx$	. . . . A
2.	$\neg \exists x \neg Fx$	$\vdash \neg \exists x \neg Fx$	. . . . A
3.		$\vdash \neg \exists x \neg Fx \supset \forall x Fx$	. . . . QE
4.	$\neg \exists x \neg Fx$	$\vdash \forall x Fx$	. . . . 2,3,⊃E
5.	$\neg \forall x Fx, \neg \exists x \neg Fx$	$\vdash \forall x Fx$	 1
6.	$\neg \forall x Fx$	$\vdash \neg \neg \exists x \neg Fx$	. . . . 4,5,¬I
7.	$\neg \forall x Fx$	$\vdash \exists x \neg Fx$	. . . . 6,¬E

(c) $\forall x \neg Fx \vdash \neg \exists x Fx$. Prove this.

1.	$\forall x \neg Fx$	$\vdash \forall x \neg Fx$	. . . . A
2.	$\exists x Fx$	$\vdash \exists x Fx$	. . . . A
3.		$\vdash \exists x Fx \supset \neg \forall x \neg Fx$	. . . . QE
4.	$\exists x Fx$	$\vdash \neg \forall x \neg Fx$	. . . . 2,3,⊃E
5.	$\forall x \neg Fx, \exists x Fx$	$\vdash \forall x \neg Fx$	 1
6.	$\forall x \neg Fx$	$\vdash \neg \exists x Fx$	. . . . 4,5,¬I

(d) $\neg\exists x Fx \vdash \forall x \neg Fx$. Prove this.

1.	$\neg\exists x Fx$	$\vdash \neg\exists x Fx$	A
2.	$\neg\forall x\neg Fx$	$\vdash \neg\forall x\neg Fx$	A
3.		$\vdash \neg\forall x\neg Fx \supset \exists x Fx$	QE
4.	$\neg\forall x\neg Fx$	$\vdash \exists x Fx$	2,3,$\supset$E
5.	$\neg\exists x Fx, \neg\forall x\neg Fx$	$\vdash \neg\exists x Fx$	1
6.	$\neg\exists x Fx$	$\vdash \neg\neg\forall x\neg Fx$	4,5,$\neg$I
7.	$\neg\exists x Fx$	$\vdash \forall x\neg Fx$	6,$\neg$E

4. Let's see some cases of interactions between the conditional and quantifiers. (Hint for constructing proofs in this section: The proofs have a lot in common).

   (a) $\forall xFx \supset \forall xGx \vdash \exists x\neg Fx \vee \forall xGx$. Add the missing annotations:

1.	$\forall xFx \supset \forall xGx$	$\vdash$	$\forall xFx \supset \forall xGx$	A
2.		$\vdash$	$(\forall xFx \supset \forall xGx) \supset (\neg\forall xFx \vee \forall xGx)$	IM
3.	$\forall xFx \supset \forall xGx$	$\vdash$	$\neg\forall xFx \vee \forall xGx$	1,2,⊃E
4.	$\neg\forall xFx$	$\vdash$	$\neg\forall xFx$	A
5.		$\vdash$	$\neg\forall xFx \supset \exists x\neg Fx$	QE
6.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx$	4,5,⊃E
7.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	6, ∨I
8.	$\forall xGx$	$\vdash$	$\forall xGx$	A
9.	$\forall xGx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	8, ∨I
10.	$\forall xFx \supset \forall xGx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	3,7,9,∨E

   (b) $\forall xFx \supset \exists xGx \vdash \exists x\neg Fx \vee \exists xGx$. Prove this.

1.	$\forall xFx \supset \exists xGx$	$\vdash$	$\forall xFx \supset \exists xGx$	A
2.		$\vdash$	$(\forall xFx \supset \exists xGx) \supset (\neg\forall xFx \vee \exists xGx)$	IM
3.	$\forall xFx \supset \exists xGx$	$\vdash$	$\neg\forall xFx \vee \exists xGx$	1,2,⊃E
4.	$\neg\forall xFx$	$\vdash$	$\neg\forall xFx$	A
5.		$\vdash$	$\neg\forall xFx \supset \exists x\neg Fx$	QE
6.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx$	4,5,⊃E
7.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx \vee \exists xGx$	6,∨I
8.	$\exists xGx$	$\vdash$	$\exists xGx$	A
9.	$\exists xGx$	$\vdash$	$\exists x\neg Fx \vee \exists xGx$	8,∨I
10.	$\exists xFx \supset \exists xGx$	$\vdash$	$\exists x\neg Fx \vee \exists xGx$	3,7,9,∨E

(c) $\exists xFx \supset \forall xGx \vdash \forall x\neg Fx \vee \forall xGx$. Prove this.

1.	$\exists xFx \supset \forall xGx$	$\vdash \exists xFx \supset \forall xGx$	A
2.		$\vdash (\exists xFx \supset \forall xGx) \supset (\neg\exists xFx \vee \forall xGx)$	IM
3.	$\exists xFx \supset \forall xGx$	$\vdash \neg\exists xFx \vee \forall xGx$	1,2,⊃E
4.	$\neg\exists xFx$	$\vdash \neg\exists xFx$	A
5.		$\vdash \neg\exists xFx \supset \forall x\neg Fx$	QE
6.	$\neg\exists xFx$	$\vdash \forall x\neg Fx$	4,5,⊃E
7.	$\neg\exists xFx$	$\vdash \forall x\neg Fx \vee \forall xGx$	6,∨I
8.	$\forall xGx$	$\vdash \forall xGx$	A
9.	$\forall xGx$	$\vdash \forall x\neg Fx \vee \forall xGx$	8,∨I
10.	$\exists xFx \supset \forall xGx$	$\vdash \forall x\neg Fx \vee \forall xGx$	3,7,9,∨E

(d) $\exists xFx \supset \exists xGx \vdash \forall x\neg Fx \vee \exists xGx$. Prove this.

1.	$\exists xFx \supset \exists xGx$	$\vdash \exists xFx \supset \exists xGx$	A
2.		$\vdash (\exists xFx \supset \exists xGx) \supset (\neg\exists xFx \vee \exists xGx)$	IM
3.	$\exists xFx \supset \exists xGx$	$\vdash \neg\exists xFx \vee \exists xGx$	1,2,⊃E
4.	$\neg\exists xFx$	$\vdash \neg\exists xFx$	A
5.		$\vdash \neg\exists xFx \supset \forall x\neg Fx$	QE
6.	$\neg\exists xFx$	$\vdash \forall x\neg Fx$	4,5,⊃E
7.	$\neg\exists xFx$	$\vdash \forall x\neg Fx \vee \exists xGx$	6,∨I
8.	$\exists xGx$	$\vdash \exists xGx$	A
9.	$\exists xGx$	$\vdash \forall x\neg Fx \vee \exists xGx$	8,∨I
10.	$\exists xFx \supset \exists xGx$	$\vdash \forall x\neg Fx \vee \exists xGx$	3,7,9,∨E

5. Brackets matter.

(a) Prove $\exists x(Fx \supset \forall yGy) \vdash \exists x \neg Fx \vee \forall xGx$. (Just so you see the importance of brackets, compare this to 3c.)

1.	$\exists x(Fx \supset \forall yGy)$	$\vdash$	$\exists x(Fx \supset \forall yGy)$	A
2.	$Fa \supset \forall yGy$	$\vdash$	$Fa \supset \forall yGy$	A
3.		$\vdash$	$(Fa \supset \forall yGy) \supset (\neg Fa \vee \forall yGy)$	IM
4.	$Fa \supset \forall yGy$	$\vdash$	$\neg Fa \vee \forall yGy$	2,3,⊃E
5.	$\neg Fa$	$\vdash$	$\neg Fa$	A
6.	$\neg Fa$	$\vdash$	$\exists x \neg Fx$	5,∃I
7.	$\neg Fa$	$\vdash$	$\exists x \neg Fx \vee \forall xGx$	6,∨I
8.	$\forall yGy$	$\vdash$	$\forall yGy$	A
9.	$\forall yGy$	$\vdash$	Ga	8,∀E
10.	$\forall yGy$	$\vdash$	$\forall xGx$	9,∀I
11.	$\forall yGy$	$\vdash$	$\exists x \neg Fx \vee \forall xGx$	10,∨I
12.	$Fa \supset \forall yGy$	$\vdash$	$\exists x \neg Fx \vee \forall xGx$	4,7,11,∨E
13.	$\exists x(Fx \supset \forall yGy)$	$\vdash$	$\exists x \neg Fx \vee \forall xGx$	1,12,∃E

6. Construct models that fit the given specifications:

(a) $\exists x F x$ true, $\forall x G x$ true.

Domain of discourse: Ash, Beck. Referent of constants: a refers to Ash, b refers to Beck. Extension of F: a. Extension of G: a and b.

(b) $\exists x F x$ true, $\forall x G x$ false.

Domain of discourse: Ash, Beck. Referent of constants: a refers to Ash, b refers to Beck. Extension of F: a. Extension of G: b.

(c) $\exists x F x$ false, $\forall x G x$ true.

Domain of discourse: Ash, Beck. Referent of constants: a refers to Ash, b refers to Beck. Extension of F: empty. Extension of G: a and b.

(d) $\exists x F x \supset \forall x G x$ false.

same as (b)

7. Explain why it is not possible to construct a model that fits the given specifications?

   (a) $\forall x(Fx \supset Gx)$ true, $\forall x(\neg Fx \vee Gx)$ false.

   > $s_1 \supset s_2$ and $\neg s_1 \vee s_2$ are logically equivalent. So for any constant κ, $F\kappa \supset G\kappa$ is true iff. $\neg F\kappa \vee G\kappa$. So we cannot make $\forall x(Fx \supset Gx)$ true without making $\forall x(\neg Fx \vee Gx)$ also true.

   (b) $\forall x(Fx \supset Gx)$ true, $\exists xFx$ true, $\exists xGx$ false.

   > If the first sentence is true, then anything that is F is also G. So if there is something that is F as required by the truth of the second sentence, then the third sentence, $\exists xGx$, must also be true.

   (c) $\exists xFx$ false, $\exists xGx$ false, $\exists x(Fx \vee Gx)$ true.

   > If anything is such that it is F or G as demanded by the third sentence, it must be F or G. So at least one of $\exists xFx$ and $\exists xGx$ must be true.